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RG/Padé estimate of the three-loop contribution to the QCD static potential function

F.A. Chishtie [1], V. Elias [1]

Theory Group, KEK, Tsukuba, Ibaraki 305-0801, Japan

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Abstract

The three renormalization-group-accessible three-loop coefficients of powers of logarithms within the $\overline{\text{MS}}$ series momentum-space for the QCD static potential are calculated and compared to values obtained via asymptotic Padé-approximant methods. The leading and next-to-leading logarithmic coefficients are both found to be in *exact* agreement with their asymptotic Padé-predictions. The predicted value for the third RG-accessible coefficient is found to be within 7% relative |error| of its true value for $n_f \leqslant 6$, and is shown to be in exact agreement with its true value in the $n_f \to \infty$ limit. Asymptotic Padé estimates are also obtained for the remaining (RG-inaccessible) three-loop coefficient. Comparison is also made with recent estimates of the three-loop contribution to the configuration-space static-potential function.

The perturbative portion of the QCD static potential is presently known to two subleading orders of perturbation theory [1–3]. This potential may be expressed as an integral over an $\overline{\text{MS}}$ perturbative-QCD series in momentum space,

$$V_{\text{pert}}(r) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(-\frac{16\pi^2}{3\vec{q}^2}\right) \times W\left[x(\mu), L\left(\mu, \vec{q}^2\right)\right], \tag{1}$$

where

$$x(\mu) \equiv \alpha_s(\mu)/\pi, \qquad L\left(\mu, \vec{q}^2\right) \equiv \log\left(\mu^2/\vec{q}^2\right), \tag{2}$$

and where the momentum-space series within the integrand of (1) is of the form

$$W[x, L] = x\left[1 + (a_0 + a_1 L)x + \left(b_0 + b_1 L + b_2 L^2\right)x^2 + \left(c_0 + c_1 L + c_2 L^2 + c_3 L^3\right)x^3 + \cdots\right] \tag{3}$$

with the following known coefficients [1]:

$$a_0 = 31/12 - 5n_f/18, \qquad a_1 = 11/4 - n_f/6, \tag{4a}$$

$$b_0 = 28.5468 - 4.14714 n_f + 25n_f^2/324, \tag{4b}$$

$$b_1 = 247/12 - 229n_f/72 + 5n_f^2/54, \tag{4c}$$

$$b_2 = 121/16 - 11n_f/12 + n_f^2/36. \tag{4d}$$

The three-loop order momentum-space coefficients c_k have not been calculated.[2] It should be noted, however, that the series' convergence may be problematical for values of x near 0.1; e.g., if $n_f = 3$ and $\mu^2 =$

E-mail address: velias@uwo.ca (V. Elias).
[1] Permanent address: Department of Applied Mathematics, The University of Western Ontario, London, Ontario N6A 5B7, Canada.

[2] A leading-log three-loop contribution in configuration space is extracted in Refs. [4].

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$\vec{q}^{\,2}$, $W[x,0] = x(1 + 1.75x + 16.80x^2 + \cdots]$. There is clearly phenomenological value in having some knowledge of the next-order coefficients c_k within the series (3), even if one chooses sufficiently large values of μ to ensure that the expansion parameter $x(\mu)$ remains small.

The all-orders momentum space static potential should ultimately be independent of the $\overline{\text{MS}}$ renormalization parameter μ:

$$\mu^2 \frac{d}{d\mu^2} W\big[x(\mu), L\big(\mu, \vec{q}^{\,2}\big)\big] = 0. \tag{5}$$

Eq. (5) corresponds to the following renormalization-group (RG) equation for the series $W[x,L]$:

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x}\right) W[x,L] = 0 \tag{6}$$

with [5]

$$\beta(x) \equiv \mu^2 \frac{d}{d\mu^2} x(\mu) = -\sum_{k=0}^{\infty} \beta_k x^{k+2}, \tag{7}$$

$$\beta_0 = 11/4 - n_f/6, \tag{8a}$$

$$\beta_1 = 51/8 - 19n_f/24, \tag{8b}$$

$$\beta_2 = 2857/128 - 5033n_f/1152 + 325n_f^2/3456. \tag{8c}$$

To leading and next-to-leading orders in perturbation theory, Eq. (6) is manifestly satisfied by the known coefficients (4) for the perturbative series (3):

$$\begin{aligned} 0 = \left(\frac{\partial}{\partial L} + \beta\frac{\partial}{\partial x}\right) W[x,L] &= (a_1 - \beta_0)x^2 \\ &+ (b_1 - 2a_0\beta_0 - \beta_1)x^3 + (2b_2 - 2\beta_0 a_1)x^3 L \\ &+ \mathcal{O}\big(x^4, x^4 L, x^4 L^2\big). \end{aligned} \tag{9}$$

The coefficients of x^2, x^3 and $x^3 L$ in (9) are all seen to vanish for known series coefficients (4) and β-function coefficients (8): i.e., the known values of b_1 and b_2 are seen to uphold the RG-equation (6) by satisfying its perturbative formulation (9). However, it is important to note that (9) may also be utilized to extract all but one of the three-loop coefficients c_k in the series (3). The coefficients of $x^4, x^4 L$ and $x^4 L^2$ in (9), respectively, vanish provided

$$\begin{aligned} c_1 = 290.769 &- 60.4881n_f \\ &+ 3.2440n_f^2 - 25n_f^3/648, \end{aligned} \tag{10}$$

$$\begin{aligned} c_2 = 1639/16 &- 4129n_f/192 \\ &+ 377n_f^2/288 - 5n_f^3/216, \end{aligned} \tag{11}$$

$$\begin{aligned} c_3 = (11/4)^3 &- 121n_f/32 \\ &+ 11n_f^2/48 - n_f^3/216. \end{aligned} \tag{12}$$

Consequently, the only RG-inaccessible three-loop-order term in the series (3) is c_0. This coefficient can be obtained only via a direct perturbative calculation, which has not yet been performed.

In the absence of such a three-loop calculation, we employ the anticipated error of Padé approximants in predicting next-order terms of a field theoretical series in order to obtain an estimate of all four three-loop coefficients c_k within (3). The predictions for $\{c_1, c_2, c_3\}$ can then be compared to their true values (10)–(12) to check the validity of the estimation procedure. The procedure we describe below has already been employed in a large number of applications: QCD β- and γ-functions [6–8], the SQCD β-function [7,9], QCD current-correlation functions [8,10], the renormalization-group functions of O(N)-symmetric massive scalar field theory [6,8, 11], Higgs decays [8,10,12], Higgs-mediated scattering processes [13], and QCD corrections to inclusive semileptonic B-decays [14]. The general method we employ is described in Refs. [7,8] and [14]; we restate its development here for convenience.

Consider a perturbative series

$$\begin{aligned} W(x) = 1 &+ R_1 x + R_2 x^2 \\ &+ R_3 x^3 + \cdots + R_N x^N + \cdots. \end{aligned} \tag{13}$$

For many such series, the series sum can be approximated by an $[N|M]$ Padé-approximant, where N and M are, respectively, the degrees of numerator and denominator polynomials within the approximant. If only the next-to-leading term R_1 is known, for example, the Padé approximant

$$W^{[0|1]}(x) = \frac{1}{1 - R_1 x} = 1 + R_1 x + R_1^2 x^2 + \cdots \tag{14}$$

would predict a value of R_1^2 for the coefficient R_2 in (13). Somewhat more realistically, if R_1 and R_2 in (13) are both known, the Padé approximant

$$\begin{aligned} W^{[1|1]}(x) &= \frac{1 + (R_1 - R_2/R_1)x}{1 - (R_2/R_1)x} \\ &= 1 + R_1 x + R_2 x^2 + (R_2^2/R_1)x^3 + \cdots \end{aligned} \tag{15}$$

leads to the predicted value R_2^2/R_1 for the coefficient R_3 in (13). Generally, one finds that the higher the degree of the approximant, the more accurate the prediction of the next unknown coefficient of the series will be. Suppose one now utilises an $[N-1|1]$ approximant to estimate the coefficient R_{N+1} within (13), based upon knowledge of all previous series coefficients $\{R_1, R_2, \ldots, R_N\}$. For perturbative field-theoretical series, it is often found that the relative error in such an estimate is inversely proportional to N [6,7,15]:

$$\left(R_{N+1}^{\text{pred}} - R_{N+1}^{\text{true}}\right)\big/R_{N+1}^{\text{true}} \cong -A/N. \tag{16}$$

The constant A in (16) can be estimated by comparing the [0|1]-approximant estimate for R_2 (i.e., $R_2^{\text{pred}} = R_1^2$) against R_2's true value. One then finds via (16) that

$$A \cong 1 - R_1^2/R_2. \tag{17}$$

In the series (13), let us suppose we only know the subleading and NNLO coefficients R_1 and R_2, as is the case for the series (3). If the [1|1] approximant prediction $R_3^{\text{pred}} = R_2^2/R_1$ has a relative error described by (16), we can substitute (17) into (16) to obtain the "true" value for R_3 *algebraically* [8,14]:

$$R_3^{\text{true}} \cong \frac{R_2^2/R_1}{1-A/2} = \frac{2R_2^3}{R_1^3 + R_1 R_2}. \tag{18}$$

Of course, the validity of this result can be ascertained only by seeing how well it predicts coefficients that can be extracted by other means. [3] For the case of the series (3), we identify the known coefficients R_1 and R_2 as polynomials in the logarithm L [1]:

$$R_1 = a_0 + a_1 L = a_0 + \beta_0 L, \tag{19}$$

$$\begin{aligned} R_2 &= b_0 + b_1 L + b_2 L^2 \\ &= b_0 + (2a_0\beta_0 + \beta_1)L + \beta_0^2 L^2. \end{aligned} \tag{20}$$

Substituting (19) and (20) into (18), we obtain the following "large-L" series expansion for R_3: [4]

$$\begin{aligned} R_3 = {} & \beta_0^3 L^3 + \left(3a_0\beta_0^2 + 5\beta_0\beta_1/2\right)L^2 \\ & + \left(a_0^2\beta_0/2 + 5\beta_0 b_0/2 \right. \\ & \quad \left. + 5a_0\beta_1/2 + 7\beta_1^2/4\beta_0\right)L^1 \\ & + \left[\beta_0^3\left(2D_1D_2 - D_1^3 - D_3\right) \right. \\ & \quad + 3\beta_0(2a_0\beta_0 + \beta_1)\left(D_1^2 - D_2\right) \\ & \quad - 3(2a_0\beta_0 + \beta_1)^2 D_1/\beta_0 - 3\beta_0 b_0 D_1 \\ & \quad + (2a_0\beta_0 + \beta_1)^3/\beta_0^3 \\ & \quad \left. + 6b_0(2a_0\beta_0 + \beta_1)/\beta_0\right]L^0 \\ & + \mathcal{O}\left(L^{-1}\right), \end{aligned} \tag{21}$$

where

$$D_1 \equiv (6a_0\beta_0 + \beta_1)/2\beta_0^2, \tag{22a}$$

$$D_2 \equiv \left[5a_0^2\beta_0 + b_0\beta_0 + a_0\beta_1\right]/2\beta_0^3, \tag{22b}$$

$$D_3 \equiv a_0\left(a_0^2 + b_0\right)/2\beta_0^3. \tag{22c}$$

As is evident from (3), R_3 should be a degree-3 polynomial in the logarithm L. A direct comparison of equivalent powers of L in (3) and in (21) leads to the following predictions:

$$c_3^{\text{pred}} = \beta_0^3, \tag{23}$$

$$c_2^{\text{pred}} = 3a_0\beta_0^2 + 5\beta_0\beta_1/2, \tag{24}$$

$$\begin{aligned} c_1^{\text{pred}} = {} & a_0^2\beta_0/2 + 5\beta_0 b_0/2 \\ & + 5\beta_1 a_0/2 + 7\beta_1^2/4\beta_0, \end{aligned} \tag{25}$$

in addition to the predicted equivalence of the *unknown* coefficient c_0 with the lengthy square-bracketed term in (21).

The prediction (23) is in exact agreement with (12), the RG-determination of c_3, as is evident by substituting (8a) into (23). Surprisingly, the predicted value (24) for c_2 is also in *exact* agreement with the RG value (11), as is evident from direct substitution of (8a), (8b) and (4a) into (24). Note that this agreement for both coefficients is true for all values of n_f, indicating that the asymptotic error formula (16) replicates the RG-invariance of the series (3) to leading and next-to-leading order in the logarithm L, a most surprising result.

[3] The formula (18), for example, is surprisingly accurate in predicting the known four-loop order β-function coefficient in O(N)-symmetric massive scalar field theory [13].

[4] Estimation of higher-order terms via such a series expansion is denoted in Ref. [10] as the "APAP′" procedure.

Table 1
Comparison of predicted and RG values for the three-loop coefficient c_1. Also displayed are predicted values for the RG-inaccessible coefficient c_0. Note that these c_0 estimates are the same sign and approximate magnitude as the RG values for c_1 listed in the second column

n_f	c_1^{RG}	c_1^{pred}	$\lvert(c_1^{\mathrm{pred}} - c_1^{\mathrm{RG}})/c_1^{\mathrm{RG}}\rvert$	c_0^{pred}
0	290.77	272	6.3%	313
1	233.49	218	6.5%	250
2	182.46	170	6.8%	193
3	137.46	128	7.0%	142
4	98.251	91.4	6.9%	97.5
5	64.606	60.6	6.2%	60.1
6	36.291	35.3	2.8%	30.5

The formula (16) cannot, of course, replicate RG invariance to all orders in L, since the infinite series (21) which follows from it is not a degree-3 polynomial in L. Nevertheless, the coefficient of L in (21) is strikingly close to the corresponding coefficient c_1 within (3), as obtained via RG-methods in (10). In Table 1, such RG determinations of c_1 are compared to the prediction (25). As is evident from the Table, the predicted values for c_1 underestimate corresponding RG values by less than 7% for $n_f \leqslant 6$, with the best agreement seen curiously to occur at $n_f = 6$. This feature may be understood by noting that the large-n_f behaviour of the estimate (25),

$$c_1 \underset{n_f \to \infty}{\longrightarrow} -25 n_f^3/648, \tag{26}$$

is in exact agreement with that of (12), the RG-determination of c_1.

Table 1 also presents estimates of the coefficient c_0, as obtained from the (square-bracketed) L^0 term in (21). This coefficient, as noted earlier, cannot be extracted from lower-order terms via RG-methods. It is nevertheless encouraging to note that corresponding predictions for c_3 and c_2 are exact, and that predictions of c_1 are nearly so. Thus, we have obtained in Table 1 asymptotic Padé-approximant estimates for the three-loop coefficient c_0, which, in conjunction with explicit RG-determinations (10)–(12) of the other three-loop coefficients $\{c_1, c_2, c_3\}$ occurring within the perturbative series $W[x, L]$ (3), constitute a prediction for the full three-loop contribution to the static-potential integrand (1).

The coordinate-space potential corresponding to the series (3) can be obtained via (1) through use of the following identities [3]:

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} = \frac{1}{4\pi r}, \tag{27a}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{\log(\mu^2/\vec{q}^2)}{\vec{q}^2} = \frac{2(\log(\mu r) + \gamma_E)}{4\pi r}, \tag{27b}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{[\log(\mu^2/\vec{q}^2)]^2}{\vec{q}^2} = \frac{4(\log(\mu r) + \gamma_E)^2 + \pi^2/3}{4\pi r}, \tag{27c}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{[\log(\mu^2/\vec{q}^2)]^3}{\vec{q}^2} = \frac{8(\log(\mu r) + \gamma_E)^3}{4\pi r} + \frac{2\pi^2(\log(\mu r) + \gamma_E) + 16\zeta(3)}{4\pi r}, \tag{27d}$$

where $\gamma_E = 0.577216$ and $\zeta(3) = 1.202057$. Following Ref. [16], we set $\mu = 1/r$ and find that

$$V_{\mathrm{pert}}(r) = \frac{\alpha_s(1/r)}{r} \sum_{n=0}^{\infty} V_n \alpha_s^n(1/r), \tag{28}$$

$$V_0 = -4/3, \tag{29a}$$

$$V_1 = -(4/3\pi)[a_0 + 2\gamma_E a_1], \tag{29b}$$

$$V_2 = -\left(4/3\pi^2\right)\left[b_0 + 2\gamma_E b_1 + \left(4\gamma_E^2 + \frac{\pi^2}{3}\right) b_2\right], \tag{29c}$$

$$V_3 = -\left(4/3\pi^3\right)\left[c_0 + 2\gamma_E c_1 + \left(4\gamma_E^2 + \frac{\pi^2}{3}\right) c_2 + \left(8\gamma_E^3 + 2\pi^2\gamma_E + 16\zeta(3)\right) c_3\right]. \tag{29d}$$

For arbitrary n_f, values of $\{a_0, a_1, b_0, b_1, b_2\}$ are given by (4), and values of $\{c_1, c_2, c_3\}$ are given by

Table 2
Configuration-space coefficients (28) of the configuration-space static potential. The column labeled V_3 is obtained using RG-determinations of c_1, c_2, c_3 and the Table 1 estimate of c_0 within Eq. (29d). The column labeled V_3^{RM} is obtained from Eq. (22) of Ref. [16]. The column labeled $V_3^{L\beta_0}$ lists large-β_0 estimates [17] that are also tabulated in Ref. [16]

n_f	V_1	V_2	V_3	V_3^{RM}	$V_3^{L\beta_0}$
3	−1.84512	−7.28304	−38.4	−37.34	−34.06
4	−1.64557	−5.94978	−28.7	−27.63	−27.03
5	−1.44602	−4.70095	−20.5	−19.46	−21.05

(10)–(12). In Table 2 we display values of the coefficients V_{1-3} obtained via (29) for $n_f = \{3, 4, 5\}$. The estimate for V_3 is obtained through use of (10)–(12) and the estimated values for c_0 in the final column of Table 1. In Table 2 we also list values of V_3 estimated via renormalon-matching (RM) considerations [16], as well as corresponding large-β_0 estimates of V_3 [17]. Striking agreement of all three estimation procedures is clearly evident in Table 2. However, it must be noted that V_3 is not very sensitive to c_0 [the only RG-inaccessible coefficient in (29d)] when $\mu = 1/r$. If one uses (29d) to extract c_0 from V_3, for example, one finds that the V_3 values −38.4, −37.34 (the RM value), and −34.06 (large β_0) tabulated in Table 2 for $n_f = 3$, respectively, correspond to c_0 values of 142 (our Table 1 RG/Padé estimate), 116, and 40.

An alternative approach to estimating c_0 follows from a least-squares fit of the asymptotic Padé-approximant prediction (18) to the three-loop momentum-space contribution's explicit dependence on L,

$$R_3 = c_0 + c_1 L + c_2 L^2 + c_3 L^3, \tag{30}$$

over the entire ultraviolet ($\mu^2 > \vec{q}^{\,2}$) region, a procedure which has been employed previously in a number of different applications [12–14]. If we define $w \equiv \vec{q}^{\,2}/\mu^2$ [i.e., $\log(w) = -L$], such a procedure entails optimization of

$$\chi^2[c_0] = \int_0^1 dw \left[\frac{2R_2^3(w)}{R_1^3(w) + R_1(w)R_2(w)} - c_0 + c_1 \log(w) - c_2 \log^2(w) + c_3 \log^3(w) \right]^2, \tag{31}$$

with respect to c_0, where

$$R_1(w) = a_0 - a_1 \log(w),$$
$$R_2(w) = b_0 - b_1 \log(w) + b_2 \log^2(w), \tag{32}$$

and where the set of known coefficients $\{a_0, a_1, b_0, b_1, b_2, c_1, c_2, c_3\}$ is given by (4) and (10)–(12). Unlike previous applications in which large-L expansions of (18) are quite consistent with least-squares fits, [5] such a fit is seen to lead to values of c_0 that are $\sim 50\%$ larger than those of Table 1:

$$n_f = 3: \quad \chi^2[c_0] = 42679 - 405.8 c_0 + c_0^2 \to c_0 = 203, \tag{33}$$

$$n_f = 4: \quad \chi^2[c_0] = 22142 - 291.1 c_0 + c_0^2 \to c_0 = 146, \tag{34}$$

$$n_f = 5: \quad \chi^2[c_0] = 9501 - 189.9 c_0 + c_0^2 \to c_0 = 95, \tag{35}$$

$$n_f = 6: \quad \chi^2[c_0] = 2901 - 104.7 c_0 + c_0^2 \to c_0 = 52. \tag{36}$$

In assessing the accuracy of (33)–(36), it should be noted that such least-squares fitting could also be employed to fit simultaneously *all four* three-loop coefficients $\{c_0, c_1, c_2, c_3\}$, as has been done before in a number of applications [12–14] in which the fitted values for $\{c_1, c_2, c_3\}$ closely approximated their known RG values. However, such a procedure completely fails for the series (3): when $n_f = 3$, optimization of (31) with respect to c_{0-3} yields values [$c_0 = 258$, $c_1 = 54.7$, $c_2 = 66.3$, $c_3 = 10.2$] that differ substantially from true values [$c_1 = 137.46$, $c_2 = 49.078$, $c_3 = 11.391$] obtained from Eqs. (10)–(12). A similarly large estimate of c_0 for the $n_f = 3$ case is obtained directly via (18) in the $L = 0$ (small-log) limit, in which $R_2 = b_0$ (4b) and $R_1 = a_0$ (4a). Such an approach yields $c_0 = 273$, a value quite comparable to that obtained above ($c_0 = 258$) by simultaneous

[5] For example, in semileptonic $b \to u$ decay, the $n_f = 4$ c_0 coefficient has a large-L-expansion value of 166, in approximate agreement with the estimate $c_0^{(4)} = 188$ obtained in [14] via (28) with known values [18] for $\{a_0, a_1, b_0, b_1, b_2\}$ and RG-determinations [14] of $\{c_1, c_2, c_3\}$ appropriate for $b \to u\ell^- \bar{\nu}_\ell$.

least-squares fitting of all four three loop coefficients c_{0-3}. If one *inputs* this small-log estimate for c_0 $(= 273)$ into (31) and then minimizes with respect to the RG-accessible coefficients c_{1-3}, the estimated values for these coefficients will be even worse than those characterising the full least squares fit ($c_1 \simeq 33$, $c_2 \simeq 74$, $c_3 \simeq 9.6$)—values which are *inconsistent* (particularly c_1) with the RG determinations of these same parameters.

Such discrepancies suggest that the large-L (i.e., large μ or short distance) c_0 estimates of Table 1, which reproduce *exact* RG values for c_2 and c_3 and closely approximate RG values for c_1, be taken more seriously than either the c_0 estimates (33)–(36) obtained via least-squares fitting over a broad range of μ, or other (e.g., small-log) approaches to estimating c_0 within an asymptotic Padé-approximant context. It is evident that such alternative asymptotic Padé-approximant estimation procedures are of little value if not tied to some way of successfully estimating c_{1-3}, the RG-accessible three loop coefficients. By this criterion, the large-L estimates of Table 1 have the most substantial credibility.

However, it is important to remain cognisant of the relative insensitivity (noted earlier) of the configuration-space static potential to the parameter c_0 at its benchmark $\mu = 1/r$ length scale. At this scale, RG-*accessible coefficients alone* dictate that the $n_f = 3$ three-loop contribution V_3 is given by

$$V_3 = -4[c_0 + 752.0]/3\pi^3, \tag{37}$$

a result which follows from substitution of (10)–(12) into (29d). Since all $n_f = 3$ estimates of c_0, as delineated in the previous two paragraphs, are small compared to 752.0, V_3 is surprisingly insensitive to this unknown parameter; the four disparate estimates 142 [large-L], 203 [Eq. (33)], 258 [full least-squares treatment], and 273 [small-log] all correspond to V_3 values near -40 [-38.4, -41.1, -43.4, and -44.1, respectively]. Thus, even if one trusts Padé methods only to give a factor-of-two-accuracy estimate for the magnitude of c_0, RG-accessible coefficients *alone* are sufficient to extract a surprisingly concise range for the three-loop contribution V_3. In other words, the $\mu = 1/r$ estimate for V_3, the three-loop quantity ultimately of phenomenological interest to us, is subject to substantially less theoretical uncertainty than the parameter c_0.

A final and necessary *caveat*, however, is the possibility that new diagrammatic topologies (and their corresponding group theoretical factors) known to enter the QCD static potential at three-loop order [4] may further circumscribe the applicability of using lower-order terms to predict the three-loop contribution c_0, as in a Padé-approximant approach. This situation is entirely analogous to the theoretically uncertain light-by-light scattering contributions known to enter the muon's anomalous magnetic moment at sufficiently high order, as well as the quartic Casimir terms first appearing in the QCD β-function series at four-loop order. Padé-approximant based techniques cannot be expected to predict terms characterised by new higher-order group-theoretical factors [7]. Nevertheless, such contributions do not necessarily dominate the first order in which they appear, nor do they necessarily devalidate Padé estimates for that order. For example, the asymptotic Padé-approximant estimate of the ($N_c = 3$) four-loop contribution to the QCD β-function [7] $\beta_3 = 23600 - 6400 n_f + 350 n_f^2 + 1.49931 n_f^3$ (estimated numbers are italicised) is in quite reasonable agreement with the exact result [19] $\beta_3 = 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3$, though in much closer agreement with the calculated result with quartic Casimir terms excised [7]: $\beta_3 = 24633 - 6375 n_f + 398.5 n_f^2 + 1.49931 n_f^3$. Thus, based on the limited information available, the remarkable success of the asymptotic Padé-approximate large-L expansion in predicting those three-loop momentum-space static potential terms that are also extractable by RG methods encourages some confidence in the corresponding large-L prediction of the RG-inaccessible parameter c_0 (modulo the above-mentioned uncertainties characterising Padé approaches), at least for purposes of predicting V_3, the three-loop contribution to the *configuration-space* static potential.

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